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Proximinal Subspaces of $C(Q)$ of Finite Codimension

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In this paper we investigate the structure of a proximinal subspace G of $C(Q)$ of codimension n, in terms of the geometry of the range of the vector measure $v = (v_1, ..., v_n)$, where $\{v_1, ..., v_n\}$ is a basis for the annihilator $G^{\perp}$. In particular, we prove that if v is non-atomic, G is proximinal iff for every $P \in \text{Ext } R(v)$ there exists a clopen subset C of $\bigcup_{i=1}^{n} S(v_i)$ such that $v(C) = P$. © 1999 Academic Press

1. INTRODUCTION

Proximinal subspaces are pivotal for best approximation in normed spaces. In addition to the general investigation of proximinality in any normed space, as in [6, 8, 17–19], there exists a large literature concerning proximinality in special cases, for instance, the case of L^p spaces discussed in [4, 9–11].

In [5] Franchetti investigated the relationship between proximinality and the existence of minimal projections for finitely complemented subspaces of a large class of Banach spaces.

In this paper we investigate finitely complemented proximinal subspaces of $C(Q)$, endowed with the usual supremum norm, with Q compact and Hausdorff. In this case the proximinality of finitely complemented subspaces has already been completely characterized by Garkavi (see the monograph [16]). A more recent characterization [20, Theorem 2] brings into the picture the existence of continuous representatives of Radon–Nikodym densities.

In this paper, using Garkavi's characterization and a Radon–Nikodym Theorem due to Greco [7], we prove that proximinal finitely complemented subspaces of $C(Q)$ enjoy the following property: *their annihilator is actually a finite-dimensional subspace of* $C(Q)$ *itself* (Theorem 3.4).

Hence in $C(Q)$ these subspaces are in a sense a surrogate of what happens in a real Hilbert space X, where X^* can be identified with X itself.

It is natural to ask whether this property characterizes proximinal subspaces among finitely complemented ones. We show that the answer is negative. This in turn leads us to a characterization of proximinality for

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2-complemented subspaces G in the particular case when $G^{\perp}$ is generated by two independent non-atomic, nonnegative measures ν_1, ν_2; namely, *G is proximinal if and only if every extreme point of the fundamental zonoid (namely the range of the measure (ν_1, ν_2)) is the image under (ν_1, ν_2) of a clopen subset of $S(\nu_1) \cup S(\nu_2)$* (Theorem 4.1). The idea of disconnectedness of the carrier in fact has already appeared in [15].

In a forthcoming paper [3] the authors obtain the same characterization for any finitely complemented subspace whose annihilator is spanned by n independent non-atomic measures and show that the non-atomicity assumption cannot be dropped in this characterization.

2. PRELIMINARIES

Throughout this paper we will adopt the following symbols:

— Q is a compact Hausdorff topological space;

— $\mathscr{B}_Q$ is the Borel σ-algebra on Q;

— $E = C(Q)$ is the space of all continuous real valued fuctions on Q, endowed with the usual supremum norm;

— $\mathbf{Q}(2)$ is the set of dyadic rational numbers;

— $R(\mu)$ is the range of a measure μ on $\mathscr{B}_Q$;

— ∂A is the boundary of a set A;

— g'_- (resp. g'_+) is the left hand side (resp. right hand side) derivative of a real function g.

For a closed subspace G of E and $x \in E$ we consider the set:

$$P_G(x) = \{ g_0 \in G : \|x - g_0\| = \min_{g \in G} \|x - g\| \}.$$

DEFINITION 2.1. If $P_G(x) \neq \emptyset$ for each $x \in E \setminus G$, then G is called a *proximinal* subspace of E.

The quotient space E/G is endowed with the norm $\|x + G\| = \operatorname{Inf}\{ \|x + y\| : y \in G\}$. As usual the dimension of E/G is termed the *codimension* of G.

DEFINITION 2.2. The set $G^{\perp} = \{ f \in E^* : f(g) = 0, \forall g \in G\}$ is called the *annihilator* of G.

Recall that, given a measure ν on $\mathscr{B}_Q$, there exists a unique compact subset $S(\nu) \subset Q$ such that $|\nu|\,(S(\nu)) = |\nu|\,(Q)$ and $|\nu|\,(H) < |\nu|\,(S(\nu))$ for every proper compact subset $H \subset S(\nu)$. Hence for every nonempty open subset A of $S(\nu)$ we have $\nu(A) > 0$. The set $S(\nu)$ is called the *carrier* of ν.

By virtue of the canonical isometry between $(E/G)^*$ and $G^\perp$, we have $\dim(G^\perp) = \dim(E/G)^*$: thus if $\operatorname{codim}(G) = n$, then $\dim(G^\perp) = n$. Hence $G^\perp = \operatorname{span}\{\nu_1, ..., \nu_n\}$, for some regular measures $\nu_1, ..., \nu_n$ on $\mathscr{B}_Q$.

The following result of Garkavi gives a characterization of the proximinal subspaces of codimension n.

THEOREM 2.1 [16]. *Let G be a closed subspace of $C(Q)$ of finite codimension. Then G is proximinal if and only if the following conditions are satisfied*:

(2.1.a) *For every $\nu \in G^\perp \backslash \{0\}$ the carrier $S(\nu)$ admits a Hahn decomposition into two disjoint closed sets $S(\nu)^+$, $S(\nu)^- = S(\nu) \backslash S(\nu)^+$,*

(2.1.b) *For every pair of measures $\nu, \bar{\nu} \in G^\perp \backslash \{0\}$ the set $S(\bar{\nu}) \backslash S(\nu)$ is closed,*

(2.1.c) *For every pair of measures $\nu, \bar{\nu} \in G^\perp \backslash \{0\}$, the measure ν is absolutely continuous with respect to $\bar{\nu}$ on the set $S(\bar{\nu})$.*

3. FINITELY COMPLEMENTED PROXIMINAL SUBSPACES OF $C(Q)$

Throughout this section G will denote a closed subspace of $C(Q)$ of codimension n; hence $G^\perp = \operatorname{span}\{\nu_1, ..., \nu_n\}$. By making use of Garkavi's Theorem, we shall prove the following:

THEOREM 3.1. *Let $\mu = |\nu_1| + \cdots + |\nu_n|$. G is proximinal if and only if for every $\nu \in G^\perp \backslash \{0\}$ there exist $f \in C(S(\mu))$ and $g \in C(S(\nu))$ such that*

$$|\nu| = \int f \, d\mu, \qquad on \quad \mathscr{B}_Q \cap S(\mu) \tag{1}$$

$$\mu = \int g \, d\nu, \qquad on \quad \mathscr{B}_Q \cap S(\nu). \tag{2}$$

Proof. In order to prove the "only if" part, we shall show that $G^\perp$ satisfies the conditions of Theorem 2.1.

We begin by proving condition (2.1.a).

Observe first that from $|\nu| = \int fg \, d\nu$ in $\mathscr{B}_Q \cap S(\nu)$ it follows

$$fg = \begin{cases} 1 & |\nu|\text{-a.e. in } S(\nu^+) \\ -1 & |\nu|\text{-a.e. in } S(\nu^-). \end{cases}$$

The sets:

$$P(v)=\{\omega\in S(v) : (fg)(\omega)=1\}$$

$$Q(v)=\{\omega\in S(v) : (fg)(\omega)=-1\}.$$

are closed, disjoint subsets of $S(v)$ such that $|v|\,(S(v^+)\backslash P(v))=0$ and $|v|(S(v^-)\backslash Q(v))=0$.

In fact, if it were $|v|\,(S(v^+)\backslash P(v))>0$, it would be $fg=1$ $|v|$-a.e. on $S(v^+)\backslash P(v)$, which contradicts the definition of $P(v)$. Likewise $|v|\,(S(v^-)\backslash Q(v))=0$. Therefore $P(v)\cup Q(v)=S(v)$ and for every $A\subset P(v)$, $v(A)\geqslant 0$, while for every $B\subset Q(v)$, $v(B)\leqslant 0$; thus $(P(v), Q(v))$ is a Hahn decomposition of $S(v)$ into two closed disjoint sets.

To prove condition (2.1.b) let us consider $v, \bar{v}\ \in G^\perp\backslash\{0\}$, and let $f, g, \bar{f}, \bar{g}$ be the corresponding continous densities of (1) and (2). We shall prove that $S(\mu)\backslash S(\bar{v})=\{\bar{f}=0\}$. Clearly $\{\bar{f}=0\}\subset S(\mu)\backslash S(\bar{v})$. To prove the converse inclusion assume that there exists $\omega\in S(\mu)\backslash S(\bar{v})$ but $\omega\notin\{\bar{f}=0\}$; then $\bar{f}(\omega)=\sigma$ (w.l.o.g. we may suppose $\sigma>0$).

Then from the continuity of $\bar{f}$ on $S(\mu)$, and being $S(\mu)\backslash\ S(\bar{v})$ an open subset of $S(\mu)$, there would exist a relative open neighbourhoood of ω, $U\subset S(\mu)\backslash S(\bar{v})$ such that $f|_U\geqslant\sigma/2$. This implies $(\sigma/2)\,\mu(U)\leqslant\int_U f\,d\mu=|\bar{v}|\,(U)=0$, and hence we would have $\mu(U)=0$. Then U would be an open nonempty subset of $S(\mu)$ with $\mu(U)=0$, which is a contradiction. Hence by the continuity of $\bar{f}$ on $S(\mu)$, we have that $S(\mu)\backslash S(\bar{v})$ is closed.

Now since $S(v)$ and $S(\bar{v})$ are both contained in $S(\mu)$ we have that $S(v)\backslash S(\bar{v})=S(v)\cap[\,S(\mu)\backslash S(\bar{v})]$. This implies that $S(v)\backslash S(\bar{v})$ is closed in $S(v)$.

The proof of (2.1.c) is straightforward.

For the proof of the "if" part, we will need the following steps.

DEFINITION 3.1. If (P, N) is a Hahn decomposition of the support of a measure $\lambda\in C(Q)^*$ into two closed disjoint sets such that $P\cup N=S(\lambda)$, we say that λ admits a *Hahn-Garkavi decomposition* (H.G.D.) (P, N).

LEMMA 3.1. *Let G be proximinal, r be a nonnegative real number and* $v\in G^\perp\backslash\{0\}$. *Then* $v^+-r\mu$ *admits a* H.G.D. *on* $S(v^+)$.

Proof. By making use of Garkavi's Theorem, we decompose $S(\mu)$ into finitely many clopen disjoint sets S_β with $\beta=(\beta_1, ..., \beta_n)$, $\beta\in\{0, 1, -1\}^n\backslash\{(0, ..., 0)\}$, according to the rule

$$S_\beta=\left[\bigcap_{\{i:\,\beta_i=1\}} S(v_i^+)\right]\cap\left[\bigcap_{\{j:\,\beta_j=-1\}} S(v_j^-)\right]\Big\backslash\left[\bigcup_{\{k:\,\beta_k=0\}} S(v_k)\right].$$

Define $\sigma_\beta=\sum_{h=1}^n \beta_h \nu_h$; then $\sigma_\beta \in G^\perp$ and in $S_\beta \cap \mathscr{B}_Q$ we have

$$\mu=\sum_{\beta_i=1} \nu_i^+ + \sum_{\beta_j=-1} \nu_j^- = \sum_{h=1}^n \beta_h \nu_h = \sigma_\beta.$$

Note that $S(\nu - r\mu) \cap S_\beta = S(\nu - r\sigma_\beta) \cap S_\beta$.

By (2.1.a), $\nu - r\sigma_\beta \in G^\perp$ admits a H.G.D. of $S(\nu - r\sigma_\beta)$, $(T_\beta{}^+, T_\beta{}^-)$. Then $(T_\beta^+ \cap S_\beta, T_\beta^- \cap S_\beta)$ is a Hahn decomposition for $\nu - r\sigma_\beta$ on $S(\nu - r\sigma_\beta) \cap S_\beta$, and from what we have already showed $(T_\beta^+ \cap S_\beta, T_\beta^- \cap S_\beta)$ is a Hahn decomposition for $\nu - r\mu$ on $S(\nu - r\mu) \cap S_\beta$.

From

$$S(\nu - r\mu) = \bigcup_\beta [S(\nu - r\mu) \cap S_\beta]$$

we have that

$$\left(\bigcup_\beta (T_\beta^+ \cap S_\beta), \bigcup_\beta (T_\beta^- \cap S_\beta)\right)$$

is a H.G.D. for $\nu - r\mu$ on $S(\nu - r\mu)$. Then $\nu - r\mu$ has a H.G.D. on $S(\nu^+) \cap S(\nu - r\mu)$. But on this set $\nu - r\mu = \nu^+ - r\mu$ and hence $\nu^+ - r\mu$ has a H.G.D. on $S(\nu^+) \cap S(\nu - r\mu)$. Note that

$$S(\nu^+) \backslash S(\nu - r\mu) = \left[\bigcap_\beta (S(\nu) \backslash S(\nu - r\sigma_\beta))\right] \cap \left[\bigcap_\beta (S(\nu) \backslash S_\beta)\right] \cap S(\nu^+).$$

By (2.1.b), $S(\nu) \backslash S(\nu - r\sigma_\beta)$ is closed for every β; and from the definition of the S_β's, $S(\nu) \cap S_\beta^c$ is a closed subset of $S(\mu)$, and hence of Q. Thus $S(\nu) \backslash [S(\nu - r\mu) \cap S_\beta]$ is closed for every β, and consequently $S(\nu^+) \backslash S(\nu - r\mu)$ is closed.

Let now (P, N) be a H.G.D. of $\nu^+ - r\mu$ on $S(\nu^+) \cap S(\nu - r\mu)$. Then $P_1 = P \cup [S(\nu^+) \backslash S(\nu - r\mu)]$ is still a positive set, and it is closed in $S(\nu^+)$. Thus (P_1, N) is a H.G.D. of $\nu^+ - r\mu$ on $S(\nu^+)$.

LEMMA 3.2. *Let G be proximinal. For every $\nu \in G^\perp$ there exists a collection of sets $(A_r)_{r \in \mathbf{Q}(2)}$ decreasing with respect to increasing r and such that, for every $r \in \mathbf{Q}(2)$, $(A_r, S(\nu^+) \backslash A_r)$ is a* H.G.D. *for $\nu^+ - r\mu$ on $S(\nu^+)$.*

Proof. From the previous lemma, there exists a collection $(A_r)_{r \in \mathbf{Q}(2)}$ of closed sets such that $(A_r, S(\nu^+) \backslash A_r)$ is a H.G.D. for $\nu^+ - r\mu$ on $S(\nu^+)$. We shall now show that $(A_r)_{r \in \mathbf{Q}(2)}$ is decreasing with respect to increasing r. Let $r \geqslant s$. From $A_r \cap A_s^c \subset A_r$, we have $\nu^+(A_r \cap A_s^c) \geqslant r\mu(A_r \cap A_s^c)$, whereas from $A_r \cap A_s^c \subset A_s^c$ it follows $\nu^+(A_r \cap A_s^c) \leqslant s\mu(A_r \cap A_s^c)$. Hence $r\mu(A_r \cap A_s^c)$

$\leqslant \nu^+(A_r \cap A_s^c) \leqslant s\mu(A_r \cap A_s^c)$ and, from $r \geqslant s$, we have $\mu(A_r \cap A_s^c) = 0$. But $\nu^+ \ll \mu$, thus $\nu^+(A_r \cap A_s^c) = 0$; since $A_r \cap A_s^c$ is open in $S(\nu^+)$, $A_r \cap A_s^c = \emptyset$. Then $A_r \subset A_s$.

The following theorem is known:

THEOREM 3.2 [7]. *Let ν, μ be two nonnegative measures on a measurable space (Ω, Σ), with $\nu \ll \mu$. Let $(A_r)_{r \in \mathbf{Q}(2)}$ be a collection of measurable subsets of Ω, decreasing with respect to increasing r and such that*

(i) $A_0 = \Omega$,

(ii) $(A_r, \Omega \setminus A_r)$ *is a Hahn decomposition for* $\nu - r\mu$,

(iii) $\lim_{r \to \infty} \nu(A_r) = 0$.

Then the function $f: \Omega \to [0, +\infty]$ defined by $f(x) = \mathrm{Sup}\{r \in \mathbf{Q}(2) : x \in A_r\}$ is such that

$$\nu(E) = \int_E f \, d\mu, \qquad \forall E \in \Sigma.$$

As a consequence of this theorem, one can establish the following:

PROPOSITION 3.1. *Let ν_1, ν_2 be two nonnegative and non-atomic measures on $\mathscr{B}_Q$, $\nu = (\nu_1, \nu_2)$, and let $F \in \mathscr{B}_Q$ be such that $\nu(F) = T \in \partial R(\nu)$. If $y = kx + \nu_2(F) - k\nu_1(F)$, $k > 0$, is a line supporting $R(\nu)$ at T, then (F, F^c) is a Hahn decomposition for $\nu_2 - k\nu_1$.*

THEOREM 3.3. *Let G be proximinal, $\nu \in G^\perp$. Then there exists a continuous function $f: S(\nu^+) \to \mathbf{R}$ such that*

$$\nu^+(E) = \int_E f \, d\mu, \qquad \forall E \in S(\nu^+) \cap \mathscr{B}_Q.$$

Proof. Let us consider the collection $(A_r)_{r \in \mathbf{Q}(2)}$ of Lemma 3.2. For $r = 0$ we have $\nu^+ - r\mu = \nu^+ \geqslant 0$ and thus $A_0 = S(\nu^+)$. Moreover, if $\nu = \sum_{i=1}^n c_i \nu_i$, for $r_0 = \max\{|c_1|, ..., |c_n|\}$ we have $\nu^+ - r_0\mu \leqslant 0$. Thus $A_s = \emptyset$ for $s \geqslant r_0$; namely, $\lim_{r \to \infty} \nu(A_r) = 0$.

The collection $(A_r)_{r \in \mathbf{Q}(2)}$ fulfills the assumptions of Theorem 3.2 with $\Omega = S(\nu^+)$; hence the function $f: S(\nu^+) \to \mathbf{R}$ defined in this theorem is a representative of the density $d\nu^+/d\mu$. Observe that f is bounded, since A_r is empty for $r \geqslant r_0$. We shall prove that f is continuous.

To show first that f is upper semicontinuous, let $t \in \mathbf{R}_0^+$ be fixed, $(x_\alpha)_\alpha \subset \{f \leqslant t\}$, with $x_\alpha \to x_0$. From $f(x_\alpha) \leqslant t$, $x_\alpha \notin A_s$, $\forall s \in \mathbf{Q}(2)$, $s > t$.

Note that the sets $S(v^+)\backslash A_s$ are closed, since we have costructed the collection $(A_r)_{r\in\mathbf{Q}(2)}$ by H.G.D. Hence $x_0 \notin A_s$, $\forall s\in\mathbf{Q}(2)$, $s>t$. Then from the definition of f we get $f(x_0)\leqslant t$.

We shall now prove that f is lower semicontinuous. Let $t\in\mathbf{R}_0^+$ and let $x_0\in\{f\geqslant t\}$: if there existed $\bar{s}<t$ with $x_0\notin A_{\bar{s}}$, from the monotonicity of the collection $(A_r)_{r\in\mathbf{Q}(2)}$, it would be $x_0\notin A_r$ $\forall r\geqslant\bar{s}$, whence

$$f(x_0)=\operatorname{Sup}\{r\in\mathbf{Q}(2): x_0\in A_r\}\leqslant\bar{s}<t\leqslant f(x_0),$$

which is a contradiction. Then $x_0\in A_s$ $\forall s<t$, and so $\{f\geqslant t\}\subset\bigcap_{s<t}A_s$. Since the converse inclusion trivially holds, $\{f\geqslant t\}$ is closed.

Applying the same technique to v^-, we obtain the following:

COROLLARY 3.1. *Let G be proximinal, and let $v\in G^{\perp}\backslash\{0\}$. Then there exist two continuous functions f, g: $S(\mu)\to\mathbf{R}$ such that*

$$v(E)=\int_E (f-g)\,d\mu, \qquad \forall E\in S(\mu)\cap\mathscr{B}_Q$$

$$|v|\,(E)=\int_E (f+g)\,d\mu, \qquad \forall E\in S(\mu)\cap\mathscr{B}_Q.$$

Proof. Let $(S(v^+), S(v^-))$ be the H.G.D. for v. Applying Theorem 3.3 we obtain two continuous functions $\bar{f}$: $S(v^+)\to\mathbf{R}$, and $\bar{g}$: $S(v^-)\to\mathbf{R}$ such that $v^+(E)=\int_E\bar{f}\,d\mu$, $\forall E\in S(v^+)\cap\mathscr{B}_Q$, and $v^-(E)=\int_E\bar{g}\,d\mu$, $\forall E\in S(v^-)\cap\mathscr{B}_Q$. Define now $f=\bar{f}\,1_{S(v^+)}$ and $g=\bar{g}\,1_{S(v^-)}$. Then

$$(f-g)(x)=\begin{cases}0 & \text{on } S(\mu)\backslash S(v)\\ f(x) & \text{on } S(\mu)\cap S(v^+)\\ -g(x) & \text{on } S(\mu)\cap S(v^-).\end{cases}$$

For every $E\in S(\mu)\cap\mathscr{B}_Q$ we have

$$\begin{aligned}\int_E (f-g)\,d\mu &= \int_{E\cap(S(\mu)\cap S(v^+))}(f-g)\,d\mu+\int_{E\cap(S(\mu)\cap S(v^-))}(f-g)\,d\mu\\ &\quad+\int_{E\cap(S(\mu)\backslash S(v))}(f-g)\,d\mu\\ &=\int_{E\cap S(v^+)}\bar{f}\,d\mu-\int_{E\cap S(v^-)}\bar{g}\,d\mu=v(E).\end{aligned}$$

Analogously $\int_E(f+g)\,d\mu=|v|\,(E)$.

The continuity of $f-g$ and $f+g$ on $S(\mu)$ is a trivial consequence of the closure of $S(v^+)$, $S(v^-)$, and $S(\mu)\backslash S(v)$. Note that this latter set is closed since

$$S(\mu)\backslash S(v)=[S(v_1)\cup\cdots\cup S(v_n)]\backslash S(v)=\left[\bigcup_{i=1} S(v_i)\backslash S(v)\right],$$

where each $S(v_i)\backslash S(v)$ is closed by (2.1.b). To conclude the proof of the "if" part of Theorem 3.1, note now that (1) follows immediately from Corollary 3.1. (2) can be proved in a completely analogous way.

As a consequence of Theorem 3.1 we have

THEOREM 3.4. *Let G be proximinal. Then there exist a regular non-negative measure μ defined on the Borel σ-algebra $\mathscr{B}_Q$, and n continuous functions $f_1, ..., f_n$: $S(\mu)\to\mathbf{R}$ such that $G^\perp\simeq \operatorname{span}\{f_1, ..., f_n\}$ (in the sense that, for every $v\in G^\perp$, there exist $\alpha_1, ..., \alpha_n\in\mathbf{R}$ such that $v(E)=\int_E(\alpha_1 f_1+\cdots+\alpha_n f_n)\,d\mu$, $\forall E\in\mathscr{B}_Q$).*

4. THE STRUCTURE OF 2-COMPLEMENTED PROXIMINAL SUBSPACES

Throughout this section G will be a closed subspace of E of codimension 2 and $G^\perp=\operatorname{span}\{v_1,v_2\}$ with v_1, v_2 non-atomic nonnegative measures on $\mathscr{B}_Q$, $S=S(v_1)\cup S(v_2)$, $v=(v_1, v_2)$. We will present a geometric characterization of proximinal subspaces of $C(Q)$, whose annihilator is spanned by two such measures. We begin with the following lemma:

LEMMA 4.1. *Let $Y=(x_Y, 0)$ and $P=(0, y_P)$ be such that*

$$x_Y=\max\{x\colon (x,0)\in\partial R(v)\}\qquad \textit{and}\qquad y_P=\max\{y\colon (0,y)\in\partial R(v)\}.$$

Let C_Y and C_P be two clopen subsets of S such that $v(C_Y)=Y$ and $v(C_P)=P$. Then $C_Y=S(v_1)\backslash S(v_2)$ and $C_P=S(v_2)\backslash S(v_1)$.

Proof. Consider the clopen set C_P. Being $v_1(C_P)=0$, we have $C_P\cap S(v_1)=\varnothing$. Then $C_P\subset S(v_2)\backslash S(v_1)$. From this inclusion and from the non-negativity of v_2 we obtain $v_2(C_P)\leqslant v_2(S(v_2)\backslash S(v_1))$. Let us suppose that C_P, which is closed and hence compact in $S(v_2)$, is strictly contained in $S(v_2)\backslash S(v_1)$. Then $C_P\cup[S(v_1)\cap S(v_2)]$ is a proper compact subset of $S(v_2)$ and thus, being $v_2(C_P)=y_P$, from the assumption we have $v_2(C_P)=v_2[S(v_2)\backslash S(v_1)]$. But $v_2(C_P\cup[S(v_1)\cap S(v_2)])=v_2(C_P)+v_2[S(v_2)\cap S(v_1)]=v_2[S(v_2)\backslash S(v_1)]+v_2[S(v_1)\cap S(v_2))]=v_2(S(v_2))$, which is a contradiction, by the definition of carrier. Then $C_P=S(v_2)\backslash S(v_1)$.

The proof of the other equality is analogous.

LEMMA 4.2. *Let* $\bar{v} = \alpha v_1 + \beta v_2$, *with* $(\alpha/\beta) < 0$ *and* $S(\bar{v}) \subset_{\neq} S$. *Then* $\partial R(v)$ *contains a segment of slope* $k = -(\alpha/\beta)$.

Proof. Throughout the following, $\forall F \in \mathscr{B}_Q$ we will denote by R_F the range of v restricted to $\mathscr{B}_F$, the Borel σ-algebra on F. Let $H = S \backslash S(\bar{v})$. Then $v_2 = kv_1$ on $\mathscr{B}_H$, and hence R_H is the segment joining the origin with $v(H)$ whose slope is in fact k. Consider the convex function g: $[0, v_1(S)] \to \mathbf{R}$ defined as

$$g(x) = \operatorname{Inf}\{v_2(U) \colon U \subset S,\, v_1(U) = x\}.$$

Set $u = \operatorname{Sup}\{t > 0 :$ there is A_t with $v_1(A_t) = t$, $v_2(A_t) = g(t)$ and $g'_-(t) < k\}$. As $R(v)$ is closed, there exists A such that $v_1(A) = u$, $v_2(A) = g(u)$ and $g'_-(u) \leqslant k$. Since R_A is contained in the cone delimited by the x-axis and the line $y = g'_-(x_A)\, x$, and as $v_1(A) = u$, we get $R_H \cap R_A = \{0\}$, namely, for every $B \subset H$, $v(A \cup B) = v(A) + v(B)$. So the set $\{v(A \cup B) : B \subset H\}$ is a segment, whose slope is k, and, by convexity, it is obvious that such segment belongs to the boundary of $R(v)$.

LEMMA 4.3. *Suppose that for every* $P \in \operatorname{Ext} R(v)$ *there exists a clopen subset* C *of* S *such that* $v(C)) = P$. *Let* $\bar{v} = \alpha v_1 + \beta v_2$ *with* $-(\alpha/\beta) = k > 0$, *be such that* $S(\bar{v}) \subset_{\neq} S$. *Then* $S(\bar{v})$ *is a clopen subset of* S.

Proof. Without loss of generality we shall prove the lemma for the measure $\tilde{v} = v_2 - kv_1$. By the previous lemma and by the central symmetry of $R(v)$, $\partial R(v)$ contains two parallel segments of slope k. Let $P = (x_P, y_P)$ and $P' = (x_{P'}, y_{P'})$, with $x_P < x_{P'}$, be the endpoints of one of those segments. Obviously $P, P' \in \operatorname{Ext} R(v)$. By assumption there exist two clopen subsets C_1, C_2 of S such that $v(C_1) = P$ and $v(C_2) = P'$.

But $C_1 \subset C_2$: in fact $C_1 \backslash C_2$ is an open subset of S. As in the proof of the H.O.B.P. [13, Lemma 3.1] one shows that $v(C_1) = v(C_1 \cap C_2)$ and thus $v(C_1 \backslash C_2) = 0$; hence $(v_1 + v_2)(C_1 \backslash C_2) = 0$. From the properties of the carrier of a measure, it follows $(C_1 \backslash C_2) = \emptyset$, that is, $C_1 \subset C_2$.

Let us now show that $K = C_2 \backslash C_1 = S \backslash S(\tilde{v})$, i.e., that $S(\tilde{v})$ is a clopen subset of S. We have $\tilde{v}(T) = 0$, $\forall T \subset K$.

Thus for $S(\tilde{v}) \backslash K$ (which is closed and hence compact in $S(\tilde{v})$) $\tilde{v}(S(\tilde{v}) \backslash K) = \tilde{v}(S(\tilde{v})) - \tilde{v}(K \cap S(\tilde{v})) = \tilde{v}(S(\tilde{v}))$.

From the properties of the carriers it follows $K \cap S(\tilde{v}) = \emptyset$, that is $S(\tilde{v}) \subset S \backslash K$. Let us suppose that $S(\tilde{v}) \subset_{\neq} S \backslash K$. This implies that $K \subset_{\neq} S \backslash S(\tilde{v})$. Then, being $|\tilde{v}|\, (S \backslash S(\tilde{v})) = 0$, it is $\tilde{v}(J) = 0$, that is $v_2(J) = kv_1(J)$ $\forall J \subset S \backslash S(\tilde{v})$. Since $K \subset_{\neq} S \backslash S(\tilde{v})$, the length of the segment of slope k on $\partial R(v)$, would be greater than that of $\overline{PP'}$. This contradicts the extremality of P', therefore $K = C_2 \backslash C_1 = S \backslash S(\tilde{v})$.

We are now able to prove the following:

THEOREM 4.1. *The following conditions are equivalent*:

(i) *G is proximinal*;

(ii) *For every $P \in \operatorname{Ext}(R(v))$ there exists a clopen subset C of S such that $v(C) = P$.*

Proof. Let us prove that (i) $\Rightarrow$ (ii).

Being v non-atomic, $R(v)$ is compact and convex [12]: hence if $P = (x_P, y_P) \in \operatorname{Ext} R(v)$, there exists a set $A \subset S$ such that $P = v(A)$. For every $A \in \mathscr{B}_Q$ define the functions

$$g_A\colon [0, v_1(A)] \to \mathbf{R}, \qquad g_A(x) = \operatorname{Inf}\{v_2(H)\colon H \subset A,\ v_1(H) = x\},$$

$$\gamma_A\colon [0, v_1(A))] \to \mathbf{R}, \qquad \gamma_A(x) = \operatorname{Sup}\{v_2(H)\colon H \subset A,\ v_1(H) = x\}.$$

When $A = S$ we shall use the symbols g and γ simply.

Define $\partial_R^- =$ graph g, and $\partial_R^+ =$ graph γ. From the geometry of zonoids, it is known that the following equality holds:

$$\partial R(v) = \partial_R^- \cup \partial_R^+ \cup \{(v_1(S), y),\ y \in [g(v_1(S)), v_2(S)]\}$$
$$\cup \{(0, y),\ y \in [0, \gamma(0)]\}.$$

Then the points $P \in \operatorname{Ext} R(v)$ such that $P \neq v(S)$ belong either to ∂_R^- or ∂_R^+. Assume for instance that $P \in \partial_R^-$, and let us consider the case: $g'_-(x_P) = g'_+(x_P) = k$. Let $P' = (x_{P'}, y_{P'}) = (v(A^c))$. It is known that A and A^c are two subsets of S forming a Hahn decomposition for the measure $v_2 - kv_1$. By (2.1.a), there exists a H.G.D. for $v_2 - kv_1$ into two clopen subsets B, C of $S(v_2 - kv_1)$. Let us suppose for example that B is the positive set and C the negative one of this decomposition. The set $D = S \backslash [S(v_2 - kv_1)]$ is clopen from condition (2.1.b) and it is of $(v_2 - kv_1)$-measure zero. Then B and C are clopen subsets of S. Then, being $(v_2 - kv_1)(D) = 0$, (B, B^c) is a H.G.D. of $v_2 - kv_1$ on S. Furthermore $v(B)$, $v(C \cup D)$, belong to $\partial R(v)$, since B and $C \cup D$ satisfy Lemma 5.2 in [14].

We shall now prove that $v(B^c) = P$. If it were $v_1(B^c) > x_P$, then, it would be $(\gamma_{B^c})'_+(0) > k$, which contradicts the negativity of B^c with respect to $v_2 - kv_1$. Likewise it cannot be $v_1(B^c) < x_P$. Then necessarily $v_1(B^c) = x_P$, and, from the extremality of P, $v_2(B^c) = y_P$.

If $g'_-(x_P) < g'_+(x_P)$, it suffices to use the same technique for the measure $v_2 - kv_1$, where k is any real number with $g'_-(x_P) < k < g'_+(x_P)$.

Let us now prove that (ii) $\Rightarrow$ (i), by means of Garkavi's Theorem. We begin by showing that every $\bar{v} = \alpha v_1 + \beta v_2 \in G^\perp \backslash \{0\}$ admits a H.G.D. The case $\alpha\beta = 0$ is trivial. Thus assume $\beta \neq 0$.

Consider first the measures $\bar{\nu}=\alpha\nu_1+\beta\nu_2$, with $(\alpha/\beta)<0$.

We have two possibilities for $S(\bar{\nu})$.

First case: $S(\bar{\nu})=S$. Since the measure $(1/\beta)\,\bar{\nu}=\nu_2+(\alpha/\beta)\,\nu_1$ has the same carrier as $\bar{\nu}$, it suffices to find a H.G.D. for $\nu_2+(\alpha/\beta)\,\nu_1$ (but the positive set with respect to $\nu_2+(\alpha/\beta)\,\nu_1$ can be the negative one with respect to $\bar{\nu}$, and conversely).

Let $x_P=\max\{x\colon (x,0)\in R(\nu)\}$, $x_{P'}=\nu_1(S(\nu_1))$. If $-(\alpha/\beta)\in[\,g'_+(x_P), g'_-(x_{P'})]$ then, being g convex, there exists at least $T\in \operatorname{Ext} R(\nu)$ such that the line $y=-(\alpha/\beta)\,x$ supports $R(\nu)$ at the point T. From the assumption there exists a clopen set F such that $\nu(F)=T$; then, by Proposition 3.1, (F, F^c) is a H.G.D. for $\nu_2+(\alpha/\beta)\,\nu_1$.

If $0<-(\alpha/\beta)<g'_+(x_P)$, take the clopen sets $S(\nu_2)\backslash S(\nu_1)$ and $S(\nu_2)$ (they are clopen by the previous lemma), while if $g'_-(x_P)<-(\alpha/\beta)<+\infty$, $(S(\nu_1)\backslash S(\nu_2), S(\nu_1))$ is the requested H.G.D.

Second case: $S(\bar{\nu})\subset_{\neq} S$. By Lemma 4.2, $\partial R(\nu)$ contains two parallel segments of slope $-(\alpha/\beta)$. Let P, P' be the endpoints of the segment belonging to ∂_R^-, and $\mathbf{V}, \mathbf{V}'$ those of the segment belonging to ∂_R^+. Necessarily either $P+\mathbf{V}$ or $P+\mathbf{V}'$ equals $\nu(S)$: let us suppose that $P+\mathbf{V}=\nu(S)$. Given that $P, \mathbf{V}\in \operatorname{Ext} R(\nu)$, by assumption (ii) there exist two disjoint clopen sets A, A^c such that $\nu(A)=\mathbf{V}$, $\nu(A^c)=P$. Then (A, A^c) is a H.G.D. for $\nu_2+(\alpha/\beta)\,\nu_1$ on S, and therefore $(A\cap S(\bar{\nu}), A^c\cap S(\bar{\nu}))$ is a H.G.D. for $\nu_2+(\alpha/\beta)\,\nu_1$.

Finally, when $(\alpha/\beta)>0$ the measure $\bar{\nu}=\nu_2+(\alpha/\beta)\,\nu_1$ is always non-negative; then $(S(\bar{\nu}), \varnothing)$ is a H.G.D. for $\bar{\nu}$ in this case.

To prove condition (2.1.b), let $\bar{\nu}=\alpha\nu_1+\beta\nu_2$, $\tilde{\nu}=\gamma\nu_1+\delta\nu_2$. We shall examine the following cases:

First case: $\alpha=\gamma=0$. Then $S(\bar{\nu})\backslash S(\tilde{\nu})=\varnothing$. Likewise if $\beta=\delta=0$.

Second case: $\beta=\gamma=0$. Then $S(\bar{\nu})\backslash S(\tilde{\nu})=S(\nu_1)\backslash S(\nu_2)$, which is a clopen set by Lemma 4.1. Likewise if $\alpha=\delta=0$.

Third case: $\alpha/\beta<0$ *and* $\gamma/\delta<0$. Then we can have

(i) $S(\tilde{\nu})=S$,

(ii) $S(\bar{\nu})=S$ and $S(\tilde{\nu}) \subsetneq S$, and

(iii) $S(\bar{\nu}) \subsetneq S$ and $S(\tilde{\nu}) \subsetneq S$.

In case (i) trivially $S(\bar{\nu})\backslash S(\tilde{\nu})=\varnothing$. In case (ii), with $S(\tilde{\nu})$ being a clopen subset of $S(\bar{\nu})=S$ (see Lemma 4.3), it turns out that $S(\bar{\nu})\backslash S(\tilde{\nu})$ is closed in Q since $S(\bar{\nu})\backslash S(\tilde{\nu})=S(\bar{\nu})\cap S(\tilde{\nu})^c$ and $S(\tilde{\nu})^c$ is closed in $S(\bar{\nu})$, and consequently in Q. In case (iii), both $S(\bar{\nu})$ and $S(\tilde{\nu})$ are clopen subsets of $S(\bar{\nu})$, and thus $S(\bar{\nu})\backslash S(\tilde{\nu})$ is closed in Q.

Fourth case: $\alpha/\beta > 0$ *and* $\gamma/\delta > 0$. Here the carrier of the measures $\bar{v}$, $\tilde{v}$ is S; then, trivially, $S(\bar{v})\backslash S(\tilde{v})$ is empty.

Fifth case: $\alpha/\beta < 0$ *and* $\gamma/\delta > 0$. In this case we have $S(\tilde{v}) = S$, and hence $S(\bar{v})\backslash S(\tilde{v})$ is empty.

Sixth case: $\alpha/\beta > 0$ *and* $\gamma/\delta < 0$. Since $S(\bar{v}) = S$, the only significant case is when $S(\tilde{v}) \subset_{\neq} S(\bar{v})$. But, by Lemma 4.3, $S(\tilde{v})$ is a clopen subset of $S(\bar{v})$, and hence $S(\bar{v})\backslash S(\tilde{v})$ is closed.

Seventh case: $\alpha, \beta \neq 0$, *and* $\delta = 0$ (*likewise when* $\alpha, \beta \neq 0$, *and* $\gamma = 0$). We have $S(\bar{v})\backslash S(\tilde{v}) = S(\bar{v})\backslash S(v_1) = S(\bar{v}) \cap S(v_1)^c = S(\bar{v}) \cap (S(v_2)\backslash S(v_1))$, which is closed from the closure of $S(v_2)\backslash S(v_1)$.

Eighth case: $\beta = 0$ *and* $\gamma, \delta \neq 0$ (*likewise when* $\alpha = 0$ *and* $\gamma, \delta \neq 0$). In this case, if $S(\tilde{v}) = S$, we have $S(\bar{v})\backslash S(\tilde{v}) = \emptyset$ while, if $S(\tilde{v}) \neq S$ by Lemma 4.3 $S(\tilde{v})$ is open in S and hence $S(\bar{v})\backslash S(\tilde{v})$ is closed.

Condition (2.1.b) is thus completely proved.

We shall prove now that for every $\bar{v}, \tilde{v} \in G^{\perp}\backslash\{0\}$, $\bar{v} \ll \tilde{v}$ on $S(\tilde{v})$.

Let us begin by showing that $v_1 \ll v_2$ on $S(v_2)$ (likewise it turns out to be $v_2 \ll v_1$ on $S(v_1)$). Since this is obvious on $S(v_2)\backslash S(v_1)$, it will suffice to prove that $v_1 \ll v_2$ on $S(v_2) \cap S(v_1)$.

If there existed $A \subset S(v_2) \cap S(v_1)$, such that $v_2(A) = 0$ but $v_1(A) > 0$, then, for the set $B = A \cup [S(v_1)\backslash S(v_2)]$ it would be $v_2(B) = v_2(A) + v_2(S(v_1)\backslash S(v_2)) = 0$, while

$$v_1(B) = v_1(A) + v_1(S(v_1)\backslash S(v_2)) > v_1(S(v_1)\backslash S(v_2)),$$

which is a contradiction by Lemma 4.1. Then $v_1 \ll v_2$ on $S(v_2)$. From this fact one can easily deduce that every $\bar{v} = \alpha v_1 + \beta v_2$ with $(\alpha, \beta) \neq (0, 0)$, is absolutely continuous with respect to $\tilde{v} = \gamma v_2$ on $S(\tilde{v})$ (likewise one can obtain $\bar{v} \ll \tilde{v} = \delta v_1$ on $S(v_1)$). Consider now $\bar{v} = \alpha v_1 + \beta v_2, (\alpha, \beta) \neq (0, 0)$ and $\tilde{v} = \gamma v_1 + \delta v_2$, $\delta \neq 0$, $-(\gamma/\delta) = \rho > 0$. Let $H \subset S(\tilde{v})$ be such that $|\tilde{v}|\,(H) = 0$. We begin by supposing that $\partial R(v)$ contains no segments of slope ρ. Then, since the range of v on H is the segment joining the origin and the point $v(H)$, if it were $v(H) \neq 0$, proceeding as in Lemma 4.2, we would obtain a linear piece of slope ρ on the boundary of $R(v)$. Then necessarily $v(H) = 0$, whence $|\bar{v}|\,(H) = 0$, that is $\bar{v} \ll \tilde{v}$ on $S(\tilde{v})$.

If on the contrary $\partial R(v)$ contains a segment of slope ρ, by hypothesis there exist two clopen subsets H_1, H_2 of S whose measures are the endpoints of the segment itself, with $H_1 \subset H_2$, and $S(\tilde{v}) = S\backslash H_3$, where $H_3 = H_2\backslash H_1$. If it were $v_1(H) > 0$, then the range of the restriction of $\tilde{v}$ to H would be a segment of slope ρ. Since $H \cap H_3 = \emptyset$, we have $H \cup H_3 \supset_{\neq} H_3$, and hence the range of the restriction of $\tilde{v}$ to $H \cup H_3$ would be a segment of slope ρ, with length greater than that of the segment

$\overline{0v(H_3)}$. Therefore $\partial R(v)$ would contain a segment of slope ρ, with length greater than that of the segment $\overline{v(H_1)v(H_2)}$, which contradicts the extremality of $v(H_2)$. Then necessarily $v_1(H)=0$ (and hence $v_2(H)=0$), whence $\bar{v} \ll \tilde{v}$ on $S(\tilde{v})$. Obviously, if $\tilde{v}=\gamma v_1+\delta v_2$, with $\gamma/\delta<0$, then $\bar{v} \ll \tilde{v}$ on $S(\tilde{v})$. The cases we have examined are sufficient to completely prove condition (2.1.c), and to conclude the proof of the theorem.

We are now able to show that the converse of Theorem 3.4 does not hold.

EXAMPLE 4.1. Let us consider the measure space $([0,1], \mathscr{B}, \lambda)$ where $\mathscr{B}$ is the Borel σ-algebra on $[0,1]$ and λ is the Lebesgue measure. Let $f(x)=x$, and define the measure $\mu(E)=\int_E x\, d\lambda$. Then clearly $S(\mu)=S(\lambda)=[0,1]$. Moreover λ and μ are non-atomic, hence $R(\lambda,\mu)$ is compact and convex. Set $G=(\text{span}\{\lambda,\mu\})^{\perp}$. Obviously, since $S(\mu)\cup S(\lambda)=[0,1]$ is connected, from Theorem 4.1, G is not proximinal.

However, the map $\mathbf{1}$ and the function $f(x)=x$, are such that $\text{span}\{\lambda,\mu\}=\{\alpha\lambda+\beta\mu : \alpha,\beta\in\mathbf{R}\}=T(\text{span}\{\mathbf{1},f\})$, where $T: h\to\int h\, d\lambda$.

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